**Functional Analysis HW 3**

Deadline: 27 Feb 2017

1. Let $X$ and $Y$ be the closed subspaces of a normed space $Z$. Show that if $\dim X < \infty$, then the sum $X + Y := \{x + y : x \in X,\ y \in Y\}$ is a closed subspace of $X$.
   **Hint**: notice that the quotient map $\pi : Z \to Z/Y$ is bounded and hence, $\pi^{-1}(F)$ is closed whenever $F$ is a closed subset of $Z/Y$.

2. Let $X = \{x \in \ell^1 : x(2k) \equiv 0, \forall k = 1, 2...\}$ and $Y = \{y \in \ell^1 : \frac{1}{k} y(2k-1) = y(2k), \forall k = 1, 2, ...\}$. For each $k = 1, 2..$, let

$$e_k(i) = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{otherwise.} \end{cases}$$

   (i) Show that $X$ and $Y$ are closed subspaces of $\ell^1$

   (ii) $e_k$ belongs the sum $X + Y$ for all $k = 1, 2...$

   (iii) $X + Y$ is not closed in $\ell^1$. **Hint**: $\overline{X + Y} = \ell^1$(Explain !). Now if we let

$$z(i) = \begin{cases} 1/n & \text{if } i = 2n, \\ 0 & \text{otherwise.} \end{cases}$$

   show that $z \notin X + Y$.

3. For each $x \in \ell^\infty$, the multiplicative linear operator $M_x : \ell^2 \to \ell^2$ defined by

$$M_x(\xi)(k) := x(k)\xi(k), \quad k = 1, 2...$$

   for $\xi \in \ell^2$. Show that $M_x$ is bounded and $\|M_x\| = \|x\|_\infty$.